Approximation of Common Fixed Points for Nonself-Asymptotically Nonexpansive Mappings

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Abstract.
Suppose that \( K \) is a nonempty closed convex subset of a real uniformly convex and smooth Banach space \( E \) with \( P \) as a sunny nonexpansive retraction. Let \( T_i : K \to E \) \((i = 1,2,3,4,5,6)\) be six of weakly inward and asymptotically nonexpansive mappings with respect to \( P \) with common sequence \( \{k_n\} \subseteq [1, \infty) \) satisfying
\[
\sum_{n=1}^{\infty} (k_n - 1) < \infty \quad \text{and} \quad \bar{F} = \bigcap_{i=1}^{6} F(T_i) \neq \emptyset, \text{respectively.}
\]
For any given \( x_1 \in K \), suppose that \( \{x_n\} \) is a sequence generated iteratively by
\[
\begin{align*}
x_{n+1} &= a_{n1} x_n + b_{n1} (PT_1)^n y_n + c_{n1} (PT_2)^n y_n,

y_n &= a_{n2} x_n + b_{n2} (PT_3)^n z_n + c_{n2} (PT_4)^n z_n,

z_n &= a_{n3} x_n + b_{n3} (PT_5)^n x_n + c_{n3} (PT_6)^n x_n,
\end{align*}
\]
where \( \{a_{ni}\}, \{b_{ni}\} \) and \( \{c_{ni}\}\) \((i = 1,2,3)\) are sequences in \([\varepsilon, 1 - \varepsilon]\) for some \( \varepsilon \in (0,1) \) which satisfy condition \( a_{ni} + b_{ni} + c_{ni} = 1 \) \((i = 1,2,3)\). Under some suitable conditions, strong convergence theorems of \( \{x_n\} \) to a common fixed point of \( \{T_i\}_{i=1}^{6} \) are obtained.

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1. Introduction
Let \( K \) be a nonempty closed convex subset of a real uniformly convex Banach space \( E \). A mapping \( T : K \to K \) is called nonexpansive if \( \|Tx - Ty\| \leq \|x - y\| \) for all \( x, y \in K \). A mapping \( T : K \to K \) is called asymptotically nonexpansive if there exists a sequence \( \{k_n\} \subseteq [1, \infty) \) with \( k_n \to 1 \) such that
\[
\|T^n x - T^n y\| \leq k_n \|x - y\| \quad (1.1)
\]
for all \( x, y \in K \) and \( n \geq 1 \).
A mapping \( T : K \to K \) is called uniformly \( L - \)Lipschitzian if there exists constant \( L \geq 0 \) such that
\[
\|T^n x - T^n y\| \leq L \|x - y\| \quad (1.2)
\]
for all \( x, y \in K \) and \( n \geq 1 \).
A subset \( K \) of \( E \) is said to be a retract of \( E \) if there exists a continuous map \( P : E \to K \) such that \( Px = x \), for all \( x \in K \). Every closed convex subset of a uniformly convex Banach space is a retract. A mapping \( P : E \to K \) is said to be a retraction if \( P^2 = P \). It follows that if a map \( P \) is a retraction, then \( Py = y \) for all \( y \in R(P) \), the range of \( P \).
It is well-known that every closed convex subset of a uniformly convex Banach space is a retract. The concept of asymptotically nonexpansive nonself-mappings was firstly introduced by Chidume et al. [2] as the generalization of asymptotically nonexpansive self-mappings. The asymptotically nonexpansive nonself-mapping is defined as follows:

Let $K$ be a nonempty subset of real normed linear space $E$. Let $P : E \rightarrow K$ be the nonexpansive retraction of $E$ onto $K$. A nonself mapping $T : K \rightarrow E$ is called asymptotically nonexpansive if there exists sequence $\{k_n\} \subseteq [1, \infty)$, $k_n \rightarrow 1$ ($n \rightarrow \infty$) such that

$$\|T(P^)-x - T(P^)-y\| \leq k_n \|x - y\|$$

for all $x, y \in K$, $n \geq 1$. (1.3)

Let $K$ be a nonempty subset of real normed linear space $E$. Let $P : E \rightarrow K$ be the nonexpansive retraction of $E$ onto $K$. A nonself mapping $T : K \rightarrow E$ is called uniformly $L$-Lipschitzian if there exists constant $L \geq 0$ such that

$$\|T(P^)-x - T(P^)-y\| \leq L \|x - y\|$$

for all $x, y \in K$, $n \geq 1$. (1.4)

As a matter of fact, if $T$ is self-mapping, then $P$ becomes the identity mapping, so that (1.3) and (1.4) reduces to (1.1) and (1.2), respectively. In addition, if $T : K \rightarrow E$ is asymptotically nonexpansive and $P : E \rightarrow K$ is a nonexpansive retraction, then $PT : K \rightarrow K$ is asymptotically nonexpansive. Indeed, for all $x, y \in K$ and $n \in \mathbb{N}$, it follows that

$$\|(P^)-x - (P^)-y\| = \|P^)-(P^)-y\|$$

$$\leq \|T(P^)-x - T(P^)-y\|$$

$$\leq k_n \|x - y\|$$

The converse, however, may not be true. Therefore, Zhou et al. [4] introduced the following generalized definition recently.

**Definition 1.** [4] Let $K$ be a nonempty subset of real normed linear space $E$. Let $P : E \rightarrow K$ be the nonexpansive retraction of $E$ into $K$.

(i) A nonself mapping $T : K \rightarrow E$ is called asymptotically nonexpansive with respect to $P$ if there exists sequences $\{k_n\} \subseteq [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\|(P^)-x - (P^)-y\| \leq k_n \|x - y\|$$

for all $x, y \in K$, $n \in \mathbb{N}$.

(ii) A nonself mapping $T : K \rightarrow E$ is said to be uniformly $L$-Lipschitzian with respect to $P$ if there exists a constant $L \geq 0$ such that

$$\|(P^)-x - (P^)-y\| \leq L \|x - y\|$$

for all $x, y \in K$, $n \in \mathbb{N}$.

Zhou et al. [4] obtained some strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings with respect to $P$ in uniformly convex Banach spaces. As a consequence, the main results of Chidume et al. [2] were deduced.

Inspired and motivated by this facts, we study three step iteration scheme for approximating common fixed points of six nonself-asymptotically nonexpansive mappings with respect to $P$ and to prove some strong convergence theorems for such mappings in uniformly convex Banach spaces. The scheme (1.5) is defined as follows.

Let $K$ be a nonempty closed convex subset of a real normed linear space $E$ with retraction $P$. Let $T_i : K \rightarrow E$ $(i = 1, 2, 3, 4, 5, 6)$ be six nonself-asymptotically nonexpansive mappings with respect to $P$. For any given $x_1 \in K$ and $n \geq 1$, suppose that $\{x_n\}$ is a sequence generated iteratively by
\[
x_{n+1} = a_{n} x_{n} + b_{n} (PT_{1})^{n} y_{n} + c_{n} (PT_{2})^{n} y_{n}, \\
y_{n} = a_{n} x_{n} + b_{n} (PT_{1})^{n} z_{n} + c_{n} (PT_{2})^{n} z_{n}, \\
z_{n} = a_{n} x_{n} + b_{n} (PT_{3})^{n} x_{n} + c_{n} (PT_{4})^{n} x_{n},
\]
(1.5)
where \( \{a_{n}\}, \{b_{n}\}, \{c_{n}\} \) \((i = 1, 2, 3)\) are sequences in \([\varepsilon, 1 - \varepsilon]\) for some \( \varepsilon \in (0, 1) \) which satisfy condition \( a_{n} + b_{n} + c_{n} = 1 \) \((i = 1, 2, 3)\).

If \( b_{n_3} = b_{n_2} = c_{n_3} = c_{n_2} = 0 \) for all \( n \geq 1 \), then (1.5) reduces to the iteration defined by Zhou et al. [4]
\[
x_1 \in K, x_{n+1} = a_{n} x_{n} + b_{n} (PT_{1})^{n} x_{n} + c_{n} (PT_{2})^{n} x_{n}, n \in \mathbb{N},
\]
(1.6)
where \( \{a_{n}\}, \{b_{n}\}, \{c_{n}\} \) are three sequences in \([\varepsilon, 1 - \varepsilon]\) for some \( \varepsilon \in (0, 1) \), satisfying \( a_{n} + b_{n} + c_{n} = 1 \).

2. Preliminaries

For the sake of convenience, we restate the following concepts and results.

Let \( E \) be a Banach space with its dimension greater than or equal to 2. The modulus of \( E \) is the function \( \delta_{E}(\varepsilon) : (0, 2] \rightarrow [0, 1] \) defined by
\[
\delta_{E}(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \left\| (x + y) \right\| : \left\| x \right\| = 1, \left\| y \right\| = 1, \varepsilon = \left\| x - y \right\| \right\}.
\]
A Banach space \( E \) is uniformly convex if and only if \( \delta_{E}(\varepsilon) > 0 \) for all \( \varepsilon \in (0, 2] \).

Let \( E \) be a Banach space and \( S(E) = \{ x \in E : \left\| x \right\| = 1 \} \). The space \( E \) is said to be smooth if
\[
\lim_{t \to 0} \frac{\left\| x + ty \right\| - \left\| x \right\|}{t}
\]
exists for all \( x, y \in S(E) \).

Let \( C \) and \( K \) be subsets of a Banach space \( E \). A mapping \( P \) from \( C \) into \( K \) is called sunny if \( P(Px + t(x - Px)) = Px \) for \( x \in C \) with \( Px + t(x - Px) \in C \) and \( t \geq 0 \).

For any \( x \in K \), the inward set \( I_{K}(x) \) is defined as follows:
\[
I_{K}(x) = \{ y \in E : y = x + \lambda(z - x), z \in K, \lambda \geq 0 \}.
\]
A mapping \( T : K \rightarrow E \) is said to satisfy the inward condition if \( Tx \in I_{K}(x) \) for all \( x \in K \). \( T \) is said to be weakly inward if \( Tx \in clI_{K}(x) \) for each \( x \in K \), where \( clI_{K}(x) \) is the closure of \( I_{K}(x) \).

A mapping \( T : K \rightarrow K \) is said to be completely continuous if for every bounded sequence \( \{x_{n}\} \), there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_{n}\} \) such that \( \{T_{x_{n_j}}\} \) converges to some element of the range \( T \).

A mapping \( T : K \rightarrow K \) is said to be demicompact if any sequence \( \{x_{n}\} \) in \( K \) satisfying \( \left\| x_{n} - T_{x_{n}} \right\| \to 0 \) as \( n \to \infty \) has a convergent subsequence.

We need the following lemmas for our main results.

**Lemma 1.**[5] If \( \{r_{n}\}, \{t_{n}\} \) are two sequences of nonnegative real numbers such that
\[
r_{n+1} = (1 + t_{n})r_{n}, \quad n \geq 1
\]
and \( \sum_{n=1}^{\infty} t_{n} < \infty \), then \( \lim_{n \to \infty} r_{n} \) exists.

**Lemma 2.**[3] Let \( E \) be real smooth Banach space, let \( K \) be nonempty closed convex subset of \( E \) with \( P \) as a sunny nonexpansive retraction, and let \( T : K \rightarrow E \) be a mapping satisfying weakly inward condition. Then \( F(PT) = F(T) \).
Lemma 3. [1] Let $E$ be a real uniformly convex Banach space and $B_R = \{ x \in E : \| x \| \leq R \}$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$, $g(0) = 0$ such that
$$\| \alpha x + \beta y + \gamma z \| \leq \alpha \| x \|^2 + \beta \| y \|^2 + \gamma \| z \|^2 - \alpha \beta \gamma g(\| x - y \|),$$
for all $x, y, z \in B_R$, and all $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

3. Main Results

In this section, we present some several strong convergence theorems of the three step iteration scheme (1.5) to a common fixed point for six nonself-asymptotically nonexpansive mappings with respect to $P$ in a real uniformly convex Banach spaces. We shall make use of the following lemmas.

Lemma 4. Let $E$ be a real normed space and $K$ a nonempty closed convex subset of $E$ which is also a nonexpansive retract of $E$. Let $T_i : K \rightarrow E$ $(i = 1, 2, 3, 4, 5, 6)$ be six nonself-asymptotically nonexpansive mappings with respect to $P$ with common sequence $\{ k_n \} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose that $\{ x_n \}$ is defined by (1.5). If $\overline{F} = \bigcap_{i=1}^{6} P(T_i) \neq \emptyset$, then $\lim_{n \rightarrow \infty} \| x_n - p \|$ exists for all $p \in \overline{F}$.

Proof. Let $p \in \overline{F}$. From (1.5), we have
$$\| z_n - p \| = \left\| a_{n3} x_n + b_{n3} (PT_3)^{\ast} x_n + c_{n3} (PT_6)^{\ast} x_n - p \right\| \leq a_{n3} \| x_n - p \| + b_{n3} k_n \| x_n - p \| + c_{n3} k_n \| x_n - p \| \leq k_n \| x_n - p \|.$$ 

By (1.5) and (3.1), we obtain
$$\| y_n - p \| = \left\| a_{n2} x_n + b_{n2} (PT_3)^{\ast} z_n + c_{n2} (PT_6)^{\ast} z_n - p \right\| \leq a_{n2} \| x_n - p \| + b_{n2} k_n \| z_n - p \| + c_{n2} k_n \| z_n - p \| \leq a_{n2} \| x_n - p \| + b_{n2} k_n \| x_n - p \| + c_{n2} k_n \| x_n - p \| \leq k_n \| x_n - p \|.$$ 

Therefore, it follows from (1.5) and (3.2) that
$$\| x_{n+1} - p \| = \left\| a_{n1} (x_n - p) + b_{n1} (PT_1)^{\ast} y_n - p \right\| + c_{n1} (PT_2)^{\ast} y_n - p \| \leq a_{n1} \| x_n - p \| + b_{n1} k_n \| y_n - p \| + c_{n1} k_n \| y_n - p \| \leq a_{n1} \| x_n - p \| + b_{n1} k_n \| x_n - p \| + c_{n1} k_n \| x_n - p \| \leq k_n \| x_n - p \| \leq (1 + \theta_n) \| x_n - p \|.$$ 

Note that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ is equivalent to $\sum_{n=1}^{\infty} (k_n^3 - 1) < \infty$. Thus, by (3.3) and Lemma 1, $\lim_{n \rightarrow \infty} \| x_n - p \|$ exists for all $p \in \overline{F}$. This completes the proof.

Lemma 5. Let $E$ be a real uniformly convex Banach space and $K$ a nonempty closed convex subset of $E$ which is also a nonexpansive retract of $E$. Let $T_i : K \rightarrow E$ $(i = 1, 2, 3, 4, 5, 6)$ be six nonself-asymptotically nonexpansive mappings with respect to $P$ with common sequence $\{ k_n \} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. 


Suppose that \( \{x_n\} \) is defined by (1.5), where \( \{a_{ni}\}, \{b_{ni}\} \) and \( \{c_{ni}\} \) \( (i = 1, 2, 3) \) are sequences in \( [\varepsilon, 1 - \varepsilon] \) for some \( \varepsilon \in (0, 1) \). If \( F = \bigcap_{i=1}^6 F(T_i) \neq \emptyset \), then \( \lim_{n \to \infty} \|x_n - (PT_1)x_n\| = 0 \) for each \( i = 1, 2, 3, 4, 5, 6 \).

**Proof.** From (1.5), by the property of \( P \), and Lemma 3, we have

\[
\|z_n - p\|^2 = \left\|a_{n3}x_n + b_{n3}(PT_3)^{\gamma}x_n + c_{n3}(PT_6)^{\gamma}x_n - p\right\|^2
\]

\[
\leq a_{n3}\|x_n - p\|^2 + b_{n3}\left\|(PT_3)^{\gamma}x_n - p\right\|^2 + c_{n3}\left\|(PT_6)^{\gamma}x_n - p\right\|^2
\]

\[
- a_{n3}b_{n3}g_1\left\|(x_n - (PT_3)^{\gamma}x_n)\right\|
\]

\[
\leq a_{n3}\|x_n - p\|^2 + b_{n3}\|k_3^n\|\|x_n - p\|^2 + c_{n3}\|k_3^n\|\|x_n - p\|^2
\]

\[
- \varepsilon^2 g_1\left\|(x_n - (PT_3)^{\gamma}x_n)\right\|
\]

\[
\leq k_3^n\|x_n - p\|^2 - \varepsilon^2 g_1\left\|(x_n - (PT_3)^{\gamma}x_n)\right\|
\]

which implies that \( g_1\left\|(x_n - (PT_3)^{\gamma}x_n)\right\| \to 0 \) as \( n \to \infty \). Since \( g_1 : [0, \infty) \to [0, \infty) \) with \( g_1(0) = 0 \) is a continuous strictly increasing convex function, it follows that

\[
\lim_{n \to \infty} \|x_n - (PT_3)^{\gamma}x_n\| = 0.
\]

Similarly, we obtain

\[
\lim_{n \to \infty} \|x_n - (PT_6)^{\gamma}x_n\| = 0.
\]

It follows from (1.5), (3.4) and Lemma 3 that

\[
\|y_n - p\|^2 = \left\|a_{n2}x_n + b_{n2}(PT_3)^{\gamma}z_n + c_{n2}(PT_4)^{\gamma}z_n - p\right\|^2
\]

\[
\leq a_{n2}\|x_n - p\|^2 + b_{n2}\left\|(PT_3)^{\gamma}z_n - p\right\|^2 + c_{n2}\left\|(PT_4)^{\gamma}z_n - p\right\|^2
\]

\[
- a_{n2}b_{n2}g_2\left\|(x_n - (PT_3)^{\gamma}z_n)\right\|
\]

\[
\leq a_{n2}\|x_n - p\|^2 + b_{n2}\|k_3^n\|\|z_n - p\|^2 + c_{n2}\|k_3^n\|\|z_n - p\|^2
\]

\[
- a_{n2}b_{n2}g_2\left\|(x_n - (PT_3)^{\gamma}z_n)\right\|
\]

\[
\leq a_{n2}\|x_n - p\|^2 + b_{n2}\|k_3^n\|\|x_n - p\|^2 + c_{n2}\|k_3^n\|\|x_n - p\|^2
\]

\[
- (b_{n2} + c_{n2})\varepsilon^2 g_2\left\|(x_n - (PT_3)^{\gamma}z_n)\right\| - a_{n2}b_{n2}g_2\left\|(x_n - (PT_3)^{\gamma}z_n)\right\|
\]

\[
\leq k_3^n\|x_n - p\|^2 - \varepsilon^2 g_2\left\|(x_n - (PT_3)^{\gamma}z_n)\right\|
\]

which implies that \( g_2\left\|(x_n - (PT_3)^{\gamma}z_n)\right\| \to 0 \) as \( n \to \infty \). Since \( g_2 : [0, \infty) \to [0, \infty) \) with \( g_2(0) = 0 \) is a continuous strictly increasing convex function, it follows that

\[
\lim_{n \to \infty} \|x_n - (PT_3)^{\gamma}z_n\| = 0.
\]

Similarly, we have

\[
\lim_{n \to \infty} \|x_n - (PT_4)^{\gamma}z_n\| = 0.
\]
Similarly, it follows from (1.5), (3.7) and Lemma 3 that

\[ \|x_{n+1} - p\|^2 = \|a_n x_n + b_n (PT_1)^y y_n + c_n (PT_2)^y y_n - p\|^2 \]
\[ \leq a_n \|x_n - p\|^2 + b_n \|(PT_1)^y y_n - p\|^2 + c_n \|(PT_2)^y y_n - p\|^2 \]
\[ - a_n b_n g_3 \left\{ x_n - (PT_1)^y y_n \right\} \]
\[ \leq a_n \|x_n - p\|^2 + b_n k_n \|y_n - p\|^2 + c_n k_n \|y_n - p\|^2 \]
\[ - (b_n + c_n) \varepsilon^2 g_3 \left\{ x_n - (PT_3)^y z_n \right\} - \varepsilon^2 g_3 \left\{ x_n - (PT_1)^y y_n \right\} \]
\[ \leq k_n \|x_n - p\|^2 - \varepsilon^2 g_3 \left\{ x_n - (PT_1)^y y_n \right\} \]

which implies that \( g_3 \left\{ x_n - (PT_1)^y y_n \right\} \rightarrow 0 \) as \( n \rightarrow \infty \). Since \( g_3 : [0, \infty) \rightarrow [0, \infty) \) with \( g_3 (0) = 0 \) is a continuous strictly increasing convex function, it follows that

\[ \lim_{n \rightarrow \infty} \|x_n - (PT_1)^y y_n\| = 0. \tag{3.10} \]

Similarly, we have

\[ \lim_{n \rightarrow \infty} \|x_n - (PT_2)^y y_n\| = 0. \tag{3.11} \]

It follows from (1.5), (3.5) and (3.6) that

\[ \|z_n - x_n\| = \|a_{n_3} x_n + b_{n_3} (PT_3)^y x_n + c_{n_3} (PT_6)^y x_n - x_n\| \tag{3.12} \]
\[ \leq b_{n_3} \|x_n - (PT_5)^y x_n\| + c_{n_3} \|x_n - (PT_6)^y x_n\| \]
\[ \rightarrow 0, \text{ as } n \rightarrow \infty \]

It follows from (3.8) and (3.12) that

\[ \{(PT_3)^y z_n - z_n\} \leq \|x_n - (PT_3)^y z_n\| + \|z_n - x_n\| \tag{3.13} \]
\[ \rightarrow 0, \text{ as } n \rightarrow \infty \]

Similarly we have

\[ \lim_{n \rightarrow \infty} \|(PT_4)^y z_n - z_n\| = 0. \tag{3.14} \]

Noting that \( y_n - z_n = a_{n_2} (x_n - z_n) + b_{n_2} (PT_3)^y z_n - z_n) + c_{n_2} (PT_3)^y z_n - z_n) \), we have

\[ \|y_n - z_n\| \leq a_{n_2} \|x_n - z_n\| + b_{n_2} \|(PT_3)^y z_n - z_n\| + c_{n_2} \|(PT_3)^y z_n - z_n\| \]

This with (3.12), (3.13) and (3.14) implies that

\[ \lim_{n \rightarrow \infty} \|y_n - z_n\| = 0 \tag{3.15} \]

From (3.12) and (3.15)

\[ \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0 \tag{3.16} \]
It follows from (3.10) and (3.16) that
\[
\left\|(PT_1)^y y_n - y_n \right\| \leq \left\|x_n - (PT_1)^y y_n \right\| + \left\|x_n - y_n \right\|
\]
\[
\rightarrow 0, \text{ as } n \rightarrow \infty
\]  
(3.17)

Similarly we have
\[
\lim_{n \rightarrow \infty} \left\|(PT_2)^y y_n - y_n \right\| = 0.
\]  
(3.18)

From (3.10) and (3.11), we have
\[
\left\|x_{n+1} - x_n \right\| = \left\|a_{n_1} x_n + b_{n_1} (PT_1)^y y_n + c_{n_1} (PT_2)^y y_n - x_n \right\|
\]
\[
\leq b_{n_1} \left\|x_n - (PT_1)^y y_n \right\| + c_{n_1} \left\|x_n - (PT_2)^y y_n \right\|
\]
\[
\rightarrow 0, \text{ as } n \rightarrow \infty
\]  
(3.19)

Noting that
\[
\left\|x_n - (PT_3)^y x_n \right\| \leq \left\|x_n - z_n \right\| + \left\|(PT_3)^y z_n - z_n \right\| + \left\||(PT_3)^y z_n - (PT_3)^y x_n \right\|
\]
\[
\leq (1 + k_n) \left\|x_n - z_n \right\| + \left\|(PT_3)^y z_n - z_n \right\|
\]
This with (3.12) and (3.13) implies that
\[
\lim_{n \rightarrow \infty} \left\|x_n - (PT_3)^y x_n \right\| = 0
\]  
(3.20)

Similarly we have
\[
\lim_{n \rightarrow \infty} \left\|x_n - (PT_4)^y x_n \right\| = 0
\]  
(3.21)

Noting that
\[
\left\|x_n - (PT_1)^y x_n \right\| \leq \left\|x_n - y_n \right\| + \left\|(PT_1)^y y_n - y_n \right\| + \left\||(PT_1)^y y_n - (PT_1)^y x_n \right\|
\]
\[
\leq (1 + k_n) \left\|x_n - y_n \right\| + \left\||(PT_1)^y y_n - y_n \right\|
\]
This with (3.16) and (3.17) implies that
\[
\lim_{n \rightarrow \infty} \left\|x_n - (PT_1)^y x_n \right\| = 0
\]  
(3.22)

Similarly we have
\[
\lim_{n \rightarrow \infty} \left\|x_n - (PT_2)^y x_n \right\| = 0.
\]  
(3.23)

Since an asymptotically nonexpansive mapping with respect to \( P \) must be uniformly \( L \)-Lipschitzian with respect to \( P \), where \( L = \sup_{n \geq 1} \{k_n\} \geq 1 \), then we have
\[
\left\|x_{n+1} - (PT_1)x_{n+1} \right\| \leq \left\|x_{n+1} - (PT_1)^{y+1} x_{n+1} \right\| + \left\|(PT_1)^{y+1} x_{n+1} - (PT_1)x_{n+1} \right\|
\]
\[
\leq \left\|x_{n+1} - (PT_1)^{y+1} x_{n+1} \right\| + L \left\|x_{n+1} - (PT_1)^y x_{n+1} \right\|
\]
\[
\leq \left\|x_{n+1} - (PT_1)^{y+1} x_{n+1} \right\| + L \left\|x_{n+1} - x_n \right\| + L \left\|x_n - (PT_1)^y x_n \right\|
\]
+ \|L(PT_i)^n x_n - (PT_i)^n x_{n+1}\|
\leq \|x_n - (PT_i)^n x_{n+1}\| + L\|x_n - (PT_i)^n x_n\|
+ L(L+1)\|x_{n+1} - x_n\|.

Consequently, by (3.5), (3.6), (3.20), (3.21), (3.22), (3.23) and (3.19), it can be obtained that
\[
\lim_{n \to \infty} \|x_n - (PT_i)x_n\| = 0 (i = 1, 2, 3, 4, 5, 6) 
\tag{3.24}
\]
This completes the proof.

**Theorem 1.** Let $E$ be a real uniformly convex Banach space and $K$ a nonempty closed convex subset of $E$. Let $T_i : K \to E (i = 1, 2, 3, 4, 5, 6)$ be six nonself-asymptotically nonexpansive mappings with respect to $P$ with common sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by (1.5), where $\{a_{ni}\}$, $\{b_{ni}\}$ and $\{c_{ni}\} (i = 1, 2, 3)$ are sequences in $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon \in (0, 1)$ and $\overline{F} = \bigcap_{i=1}^{6} (T_i) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{6}$ if and only if $\lim \inf_{n \to \infty} d(x_n, \overline{F}) = 0$, where $d(x_n, \overline{F}) = \inf \{\|x_n - p\| : p \in \overline{F}\}$.

**Proof.** The necessity is obvious. Thus, we need only prove the sufficiency. Suppose that $\lim \inf_{n \to \infty} d(x_n, \overline{F}) = 0$.

From (3.3), we obtain
\[
d(x_{n+1}, \overline{F}) \leq (1 + (k_n - 1))d(x_n, \overline{F}) 
\tag{3.25}
\]
As $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, therefore $\lim \inf_{n \to \infty} d(x_n, \overline{F})$ exists by Lemma 1. But by hypothesis $\lim \inf_{n \to \infty} d(x_n, \overline{F}) = 0$, hence we must have $\lim \inf_{n \to \infty} d(x_n, \overline{F}) = 0$. Next we shall prove that $\{x_n\}$ is a Cauchy sequence. It follows from (3.3) that for any $n, m \geq n_0$
\[
\|x_{n+m} - p\| \leq (1 + \theta_{n+m-1})\|x_{n+m-1} - p\|
\leq \exp(\theta_{n+m-1})\|x_{n+m-1} - p\|
\leq \cdots \leq \exp\left(\sum_{k=n}^{n+m-1} \theta_k\right)\|x_n - p\|
\]
Let $M = \exp\left(\sum_{k=n}^{\infty} \theta_k\right)$, then $M > 0$ and
\[
\|x_{n+m} - p\| \leq M\|x_n - p\|, \forall n, m \geq n_0. 
\tag{3.26}
\]
For an arbitrary $\varepsilon > 0$, since $\lim \inf_{n \to \infty} d(x_n, \overline{F}) = 0$, there exists a positive integer $N_1$ such that $d(x_n, \overline{F}) < \frac{\varepsilon}{4M}$ for all $n \geq N_1$. So, we have $d(x_{N_1}, \overline{F}) < \frac{\varepsilon}{4M}$. This means that there exists a $x^* \in \overline{F}$ such that
\[
\|x_{N_1} - x^*\| \leq \frac{\varepsilon}{4M}. 
\]
It follows from (3.26) that for all $n \geq N_1$ and $m \geq 1$,
\[
\|x_{n+m} - x_n\| \leq \|x_{n+m} - x^*\| + \|x_n - x^*\|
\]
58
\[
\leq 2M \|x_{N_1} - x^*\| < \varepsilon.
\]

This implies that \(\{x_n\}\) is a Cauchy sequence. Since \(E\) is complete, therefore \(\lim_{n \to \infty} x_n\) exists. Let \(\lim_{n \to \infty} x_n = q^*\). Then \(q^* \in K\). It remains to prove that \(q^* \in F\). For an arbitrary \(\varepsilon > 0\), there exists a positive integer \(N_2 > N_1\) such that \(\|x_n - q^*\| < \frac{\varepsilon}{2(1 + k_n)}\) for all \(n \geq N_2\). Since \(\lim_{n \to \infty} d(x_n, F) = 0\), there exists a natural number \(N_3 > N_2\) such that \(d(x_n, F) < \frac{\varepsilon}{2(1 + k_n)}\) for all \(n \geq N_3\). Therefore, there exists \(p^* \in F\) such that
\[
\|x_{N_3} - p^*\| < \frac{\varepsilon}{2(1 + k_n)}.
\]

Consequently we have
\[
\|PT_i q^* - q^*\| = \|PT_i q^* - p^* + (p^* - x_{N_3}) + (x_{N_3} - q^*)\|
\leq \|PT_i q^* - p^*\| + \|p^* - x_{N_3}\| + \|x_{N_3} - q^*\|
\leq k_n \|q^* - p^*\| + \|p^* - x_{N_3}\| + \|x_{N_3} - q^*\|
\leq (1 + k_n) \|p^* - x_{N_3}\| + \|x_{N_3} - q^*\|
\leq \frac{\varepsilon}{k_n}.
\]

This implies that \(q^* \in F(\{PT_i\})\). It follows from Lemma 2 that \(q^* \in F(T_i)\). Similarly, \(q^* \in F(T_2), q^* \in F(T_3), q^* \in F(T_4), q^* \in F(T_5), q^* \in F(T_6)\). Therefore \(q^* \in F\). This completes the proof.

**Theorem 2.** Let \(K\) be a nonempty closed convex subset of a real smooth and uniformly convex Banach space \(E\) with \(P\) as a sunny nonexpansive retraction. Let \(T_i : K \to E (i = 1, 2, 3, 4, 5, 6)\) be six weakly inward and nonself-asymptotically nonexpansive mappings with respect to \(P\) with common sequence \(\{k_n\} \subset [1, \infty)\) satisfying \(\sum_{n=1}^{\infty} (k_n - 1) < \infty\). Suppose that \(\{x_n\}\) is defined by (1.5), where \(\{a_n\}, \{b_n\}\) and \(\{c_n\}\) \((i = 1, 2, 3)\) are sequences in \([\varepsilon, 1 - \varepsilon]\) for some \(\varepsilon \in (0, 1)\). If one of \(\{T_{i_{j_{kl}}}\}^{6}_{i=1}\) is completely continuous and \(\bar{F} = \bigcap_{i=1}^{6} F(T_i) \neq \emptyset\), then \(\{x_n\}\) converges strongly to a common fixed point of \(\{T_{i_{j_{kl}}}\}^{6}_{i=1}\).

**Proof.** By Lemma 4 \(\lim_{n \to \infty} \|x_n - p\|\) exists for any \(p \in \bar{F}\). It is sufficient to show that \(\{x_n\}\) has a subsequence which converges strongly to a common fixed point of \(\{T_{i_{j_{kl}}}\}^{6}_{i=1}\). By Lemma 5, \(\lim_{n \to \infty} \|x_n - (PT_i)x_n\| = 0\) \((i = 1, 2, 3, 4, 5, 6)\). Suppose that \(T_{i_{j_{kl}}}\) is completely continuous. Noting that \(P\) is nonexpansive, we conclude that there exists subsequence \(\{PT_{i_{j_{kl}}}x_{n_j}\}\) of \(\{PT_i x_n\}\) such that \(PT_{i_{j_{kl}}}x_{n_j} \to p\), and hence \(x_{n_j} \to p\) as \(j \to \infty\). By the continuity of \(P, T_1, T_2, T_3, T_4, T_5\) and \(T_6\), we have \(p = PT_1 p = PT_2 p = PT_3 p = PT_4 p = PT_5 p = PT_6 p\), and so \(p \in \bar{F}\). Thus, \(\{x_n\}\) converges strongly to a common fixed point \(p\) of \(\{T_{i_{j_{kl}}}\}^{6}_{i=1}\). This completes the proof.
Theorem 3. Let $K$ be a nonempty closed convex subset of a real smooth and uniformly convex Banach space $E$ with $P$ as a sunny nonexpansive retraction. Let $T_i : K \rightarrow E$ $(i = 1, 2, 3, 4, 5, 6)$ be six weakly inward nonself-asymptotically nonexpansive mappings with respect to $P$ with common sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by (1.5), where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ $(i = 1, 2, 3)$ are sequences in $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon \in (0, 1)$, if one of $\{T_i\}_{i=1}^6$ is demicompact and $F = \bigcap_{i=1}^6 F(T_i) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^6$.

Proof. Since one of $\{T_i\}_{i=1}^6$ is demicompact, so is one of $PT_1, PT_2, PT_3, PT_4, PT_5$ and $PT_6$. Suppose that $PT_1$ is demicompact. Noting that $\{x_n\}$ is bounded, we assert that there exists a subsequence $\{PT_1x_{n_j}\}$ of $\{PT_1x_n\}$ such that $PT_1x_{n_j}$ converges strongly to $p$. By Lemma 5, we have $x_{n_j} \rightarrow p$ as $j \rightarrow \infty$. By the continuity of $P, T_1, T_2, T_3, T_4, T_5$ and $T_6$, we have $p = PT_1p = PT_2p = PT_3p = PT_4p = PT_5p = PT_6p$, and so $p \in F$ by Lemma 2. By Lemma 4, we know that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Therefore, $\{x_n\}$ converges strongly to a common fixed point $p$ as $n \rightarrow \infty$. This completes the proof.

References


