Approximation of Common Fixed Points for Nonself-Asymptotically Nonexpansive Mappings

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Abstract.

Suppose that K *is a nonempty closed convex subset of a real uniformly convex and smooth Banach space* E *with* P *as a sunny nonexpansive retraction. Let* $T_i : K \to E$ $(i = 1,2,3,4,5,6)$ *be six of weakly inward and asymptotically nonexpansive mappings with respect to* P *with common sequence* $\{k_n\} \subset [1,\infty)$ *satisfying* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $\overline{F} = \bigcap_{i=1}^{6} F(T_i) \neq \varnothing$ *, respectively. For any given* $x_1 \in K$, *suppose that* $\{x_n\}$ *is a sequence generated iteratively by*

$$\begin{cases} x_{n+1} = a_{n1}x_n + b_{n1}(PT_1)^n y_n + c_{n1}(PT_2)^n y_n, \\ y_n = a_{n2}x_n + b_{n2}(PT_3)^n z_n + c_{n2}(PT_4)^n z_n, \\ z_n = a_{n3}x_n + b_{n3}(PT_5)^n x_n + c_{n3}(PT_6)^n x_n, \end{cases}$$

where $\{a_{ni}\}, \{b_{ni}\}$ *and* $\{c_{ni}\}$ $(i = 1,2,3)$ *are sequences in* $[\varepsilon, 1-\varepsilon]$ *for some* $\varepsilon \in (0,1)$ *which satisfy condition* $a_{ni} + b_{ni} + c_{ni} = 1$ $(i = 1,2,3)$. *Under some suitable conditions, strong convergence theorems of* $\{x_n\}$ *to a common fixed point of* $\{T_i\}_{i=1}^{6}$ *are obtained.*

Keywords and phrases: Nonself-Asymptotically Nonexpansive Mappings, common fixed point, strongconvergence.

2000 Mathematics Subject Classification: 47H09, 47H10, 46B20.

1. Introduction

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. A mapping $T : K \to K$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. A mapping $T : K \to K$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in K$ and $n \geq 1$.

A mapping $T : K \to K$ is called *uniformly* $L-$*Lipschitzian* if there exists constant $L \geq 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{1.2}$$

for all $x, y \in K$ and $n \geq 1$.

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$, for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : E \to K$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all $y \in R(P)$, the range of P.

It is well-known that every closed convex subset of a uniformly convex Banach space is a retract. The concept of asymptotically nonexpansive nonself-mappings was firstly introduced by Chidume et al. [2] as the generalization of asymptotically nonexpansive self-mappings. The asymptotically nonexpansive nonself-mapping is defined as follows:

Let K be a nonempty subset of real normed linear space E. Let $P:E\to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T:K\to E$ is called *asymptotically nonexpansive* if there exists sequence $\{k_n\}\subset[1,\infty)$, $k_n\to 1\ (n\to\infty)$ such that

$$\|T(PT)^{n-1}x-T(PT)^{n-1}y\|\le k_n\|x-y\| \text{ for all } x,y\in K,\quad n\ge 1. \tag{1.3}$$

Let K be a nonempty subset of real normed linear space E. Let $P:E\to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T:K\to E$ is called *uniformly $L-$Lipschitzian* if there exists constant $L\ge 0$ such that

$$\|T(PT)^{n-1}x-T(PT)^{n-1}y\|\le L\|x-y\| \text{ for all } x,y\in K,\quad n\ge 1. \tag{1.4}$$

As a matter of fact, if T is self-mapping, then P becomes the identity mapping, so that (1.3) and (1.4) reduces to (1.1) and (1.2), respectively. In addition, if $T:K\to E$ is asymptotically nonexpansive and $P:E\to K$ is a nonexpansive retraction, then $PT:K\to K$ is asymptotically nonexpansive. Indeed, for all $x,y\in K$ and $n\in\mathbb{N}$, it follows that

$$\begin{aligned}\|(PT)^n x-(PT)^n y\| &= \|PT(PT)^{n-1}x-PT(PT)^{n-1}y\|\\ &\le \|T(PT)^{n-1}x-T(PT)^{n-1}y\|\\ &\le k_n\|x-y\|.\end{aligned}$$

The converse, however, may not be true. Therefore, Zhou et al. [4] introduced the following generalized definition recently.

Definition 1.[4] *Let K be a nonempty subset of real normed linear space E. Let $P:E\to K$ be the nonexpansive retraction of E into K.*

(i)A nonself mapping $T:K\to E$ is called asymptotically nonexpansive with respect to P if there exists sequences $\{k_n\}\in[1,\infty)$ with $k_n\to 1$ as $n\to\infty$ such that

$$\|(PT)^n x-(PT)^n y\|\le k_n\|x-y\|,\ \forall x,y\in K, n\in\mathbb{N}.$$

(ii)A nonself mapping $T:K\to E$ is said to be uniformly L-Lipschitzian with respect to P if there exists a constant $L\ge 0$ such that

$$\|(PT)^n x-(PT)^n y\|\le L\|x-y\|,\ \forall x,y\in K, n\in\mathbb{N}.$$

Zhou et al. [4] obtained some strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings with respect to P in uniformly convex Banach spaces. As a consequence, the main results of Chidume et al. [2] were deduced.

Inspired and motivated by this facts, we study three step iteration scheme for approximating common fixed points of six nonself-asymptotically nonexpansive mappings with respect to P and to prove some strong convergence theorems for such mappings in uniformly convex Banach spaces. The scheme (1.5) is defined as follows.

Let K be a nonempty closed convex subset of a real normed linear space E with retraction P. Let $T_i:K\to E$ $(i=1,2,3,4,5,6)$ be six nonself-asymptotically nonexpansive mappings with respect to P. For any given $x_1\in K$ and $n\ge 1$, suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_{n+1} = a_{n1}x_n + b_{n1}(PT_1)^n y_n + c_{n1}(PT_2)^n y_n, \\ y_n = a_{n2}x_n + b_{n2}(PT_3)^n z_n + c_{n2}(PT_4)^n z_n, \\ z_n = a_{n3}x_n + b_{n3}(PT_5)^n x_n + c_{n3}(PT_6)^n x_n, \end{cases} \tag{1.5}$$

where $\{a_{ni}\}, \{b_{ni}\}$ and $\{c_{ni}\}$ $(i = 1,2,3)$ are sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$ which satisfy condition $a_{ni} + b_{ni} + c_{ni} = 1$ $(i = 1,2,3)$.

If $b_{n3} = b_{n2} = c_{n3} = c_{n2} \equiv 0$ for all $n \geq 1$, then (1.5) reduces to the iteration defined by Zhou et al. [4]

$$x_1 \in K,\, x_{n+1} = a_{n1}x_n + b_{n1}(PT_1)^n x_n + c_{n1}(PT_2)^n x_n,\, n \in \mathrm{N}, \tag{1.6}$$

where $\{a_{n1}\}, \{b_{n1}\}$ and $\{c_{n1}\}$ are three sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$, satisfying $a_{n1} + b_{n1} + c_{n1} = 1$.

2.Preliminaries

For the sake of convenience, we restate the following concepts and results.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of E is the function $\delta_E(\varepsilon): (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x-y\|\right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Let E be a Banach space and $S(E) = \{x \in E : \|x\| = 1\}$. The space E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$.

Let C and K be subsets of a Banach space E. A mapping P from C into K is called sunny if $P(Px + t(x - Px)) = Px$ for $x \in C$ with $Px + t(x - Px) \in C$ and $t \geq 0$.

For any $x \in K$, the inward set $I_K(x)$ is defined as follows:

$$I_K(x) = \{y \in E : y = x + \lambda(z - x), z \in K, \lambda \geq 0\}.$$

A mapping $T : K \to E$ is said to satisfy the inward condition if $Tx \in I_K(x)$ for all $x \in K$. T is said to be weakly inward if $Tx \in clI_K(x)$ for each $x \in K$, where $clI_K(x)$ is the closure of $I_K(x)$.

A mapping $T : K \to K$ is said to be *completely continuous* if for every bounded sequence $\{x_n\}$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{Tx_{n_j}\}$ converges to some element of the range T.

A mapping $T : K \to K$ is said to be *demicompact* if any sequence $\{x_n\}$ in K satisfying $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ has a convergent subsequence.

We need the following lemmas for our main results.

Lemma 1.*[5] If $\{r_n\}$, $\{t_n\}$ are two sequences of nonnegative real numbers such that*

$$r_{n+1} \leq (1 + t_n) r_n, \quad n \geq 1$$

and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

Lemma 2.*[3] Let E be real smooth Banach space, let K be nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T : K \to E$ be a mapping satisfying weakly inward condition. Then* $F(PT) = F(T)$.

Lemma 3.*[1] Let E be a real uniformly convex Banach space and $B_R = \{x \in E : \|x\| \le R\}$. Then there exists a continuous, strictly increasing, and convex function $g:[0,\infty) \to [0,\infty)$, $g(0)=0$ such that*

$$\|\alpha x+\beta y+\gamma z\|^2 \le \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta g(\|x-y\|),$$

for all $x, y, z \in B_R$, and all $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma=1$.

3.Main Results

In this section, we present some several strong convergence theorems of the three step iteration scheme (1.5) to a common fixed point for six nonself-asymptotically nonexpansive mappings with respect to P in a real uniformly convex Banach spaces. We shall make use of the following lemmas.

Lemma 4.*Let E be a real normed space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_i : K \to E$ $(i=1,2,3,4,5,6)$ be six nonself-asymptotically nonexpansive mappings with respect to P with common sequence $\{k_n\} \subset [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(k_n-1)<\infty$. Suppose that $\{x_n\}$ is defined by (1.5). If $\overline{F}=\bigcap_{i=1}^{6}F(T_i)\neq\varnothing$, then $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in\overline{F}$.*

Proof. Let $p \in \overline{F}$. From (1.5), we have

$$\begin{aligned}\|z_n-p\| &= \|a_{n3}x_n+b_{n3}(PT_5)^n x_n+c_{n3}(PT_6)^n x_n-p\| \\ &\le a_{n3}\|x_n-p\|+b_{n3}k_n\|x_n-p\|+c_{n3}k_n\|x_n-p\| \\ &\le k_n\|x_n-p\|.\end{aligned} \tag{3.1}$$

By (1.5) and (3.1), we obtain

$$\begin{aligned}\|y_n-p\| &= \|a_{n2}x_n+b_{n2}(PT_3)^n z_n+c_{n2}(PT_4)^n z_n-p\| \\ &\le a_{n2}\|x_n-p\|+b_{n2}k_n\|z_n-p\|+c_{n2}k_n\|z_n-p\| \\ &\le a_{n2}\|x_n-p\|+b_{n2}k_n^2\|x_n-p\|+c_{n2}k_n^2\|x_n-p\| \\ &\le k_n^2\|x_n-p\|.\end{aligned} \tag{3.2}$$

Therefore, it follows from (1.5) and (3.2) that

$$\begin{aligned}\|x_{n+1}-p\| &= \|a_{n1}(x_n-p)+b_{n1}((PT_1)^n y_n-p)+c_{n1}((PT_2)^n y_n-p)\| \\ &\le a_{n1}\|x_n-p\|+b_{n1}k_n\|y_n-p\|+c_{n1}k_n\|y_n-p\| \\ &\le a_{n1}\|x_n-p\|+b_{n1}k_n^3\|x_n-p\|+c_{n1}k_n^3\|x_n-p\| \\ &\le k_n^3\|x_n-p\| \\ &= (1+\theta_n)\|x_n-p\|.\end{aligned} \tag{3.3}$$

Note that $\sum_{n=1}^{\infty}(k_n-1)<\infty$ is equivalent to $\sum_{n=1}^{\infty}(k_n^3-1)<\infty$. Thus, by (3.3) and Lemma 1, $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in\overline{F}$. This completes the proof.

Lemma 5.*Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_i : K \to E$ $(i=1,2,3,4,5,6)$ be six nonself-asymptotically nonexpansive mappings with respect to P with common sequence $\{k_n\} \subset [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(k_n-1)<\infty$.*

Suppose that $\{x_n\}$ *is defined by (1.5), where* $\{a_{ni}\}, \{b_{ni}\}$ *and* $\{c_{ni}\}$ $(i=1,2,3)$ *are sequences in* $[\varepsilon, 1-\varepsilon]$ *for some* $\varepsilon \in (0,1)$. *If* $\overline{F} = \bigcap_{i=1}^{6} F(T_i) \neq \emptyset$, *then* $\lim_{n\to\infty}\|x_n - (PT_i)x_n\| = 0$ *for each* $i = 1,2,3,4,5,6.$

Proof. From (1.5), by the property of P, and Lemma 3 , we have

$$\begin{aligned}\|z_n - p\|^2 &= \left\|a_{n3}x_n + b_{n3}(PT_5)^n x_n + c_{n3}(PT_6)^n x_n - p\right\|^2 \qquad (3.4)\\
&\le a_{n3}\|x_n - p\|^2 + b_{n3}\left\|(PT_5)^n x_n - p\right\|^2 + c_{n3}\left\|(PT_6)^n x_n - p\right\|^2\\
&- a_{n3}b_{n3}g_1\left(\left\|x_n - (PT_5)^n x_n\right\|\right)\\
&\le a_{n3}\|x_n - p\|^2 + b_{n3}k_n^2\|x_n - p\|^2 + c_{n3}k_n^2\|x_n - p\|^2\\
&- \varepsilon^2 g_1\left(\left\|x_n - (PT_5)^n x_n\right\|\right)\\
&\le k_n^2\|x_n - p\|^2 - \varepsilon^2 g_1\left(\left\|x_n - (PT_5)^n x_n\right\|\right)\end{aligned}$$

which implies that $g_1\left(\left\|x_n - (PT_5)^n x_n\right\|\right) \to 0$ as $n \to \infty$. Since $g_1 : [0,\infty) \to [0,\infty)$ with $g_1(0) = 0$ is a continuous strictly increasing convex function, it follows that

$$\lim_{n\to\infty}\left\|x_n - (PT_5)^n x_n\right\| = 0. \qquad (3.5)$$

Similarly, we obtain

$$\lim_{n\to\infty}\left\|x_n - (PT_6)^n x_n\right\| = 0. \qquad (3.6)$$

It follows from (1.5), (3.4) and Lemma 3 that

$$\begin{aligned}\|y_n - p\|^2 &= \left\|a_{n2}x_n + b_{n2}(PT_3)^n z_n + c_{n2}(PT_4)^n z_n - p\right\|^2 \qquad (3.7)\\
&\le a_{n2}\|x_n - p\|^2 + b_{n2}\left\|(PT_3)^n z_n - p\right\|^2 + c_{n2}\left\|(PT_4)^n z_n - p\right\|^2\\
&- a_{n2}b_{n2}g_2\left(\left\|x_n - (PT_3)^n z_n\right\|\right)\\
&\le a_{n2}\|x_n - p\|^2 + b_{n2}k_n^2\|z_n - p\|^2 + c_{n2}k_n^2\|z_n - p\|^2\\
&- a_{n2}b_{n2}g_2\left(\left\|x_n - (PT_3)^n z_n\right\|\right)\\
&\le a_{n2}\|x_n - p\|^2 + b_{n2}k_n^4\|x_n - p\|^2 + c_{n2}k_n^4\|x_n - p\|^2\\
&- (b_{n2} + c_{n2})\varepsilon^2 g_1\left(\left\|x_n - (PT_5)^n x_n\right\|\right) - a_{n2}b_{n2}g_2\left(\left\|x_n - (PT_3)^n z_n\right\|\right)\\
&\le k_n^4\|x_n - p\|^2 - \varepsilon^2 g_2\left(\left\|x_n - (PT_3)^n z_n\right\|\right),\end{aligned}$$

which implies that $g_2\left(\left\|x_n - (PT_3)^n z_n\right\|\right) \to 0$ as $n \to \infty$. Since $g_2 : [0,\infty) \to [0,\infty)$ with $g_2(0) = 0$ is a continuous strictly increasing convex function, it follows that

$$\lim_{n\to\infty}\left\|x_n - (PT_3)^n z_n\right\| = 0. \qquad (3.8)$$

Similarly, we have

$$\lim_{n\to\infty}\left\|x_n - (PT_4)^n z_n\right\| = 0. \qquad (3.9)$$

Similarly, it follows from (1.5), (3.7) and Lemma 3 that

$$\|x_{n+1}-p\|^2=\|a_{n1}x_n+b_{n1}(PT_1)^n y_n+c_{n1}(PT_2)^n y_n-p\|^2$$
$$\le a_{n1}\|x_n-p\|^2+b_{n1}\|(PT_1)^n y_n-p\|^2+c_{n1}\|(PT_2)^n y_n-p\|^2$$
$$-a_{n1}b_{n1}g_3(\|x_n-(PT_1)^n y_n\|)$$
$$\le a_{n1}\|x_n-p\|^2+b_{n1}k_n^2\|y_n-p\|^2+c_{n1}k_n^2\|y_n-p\|^2$$
$$-\varepsilon^2 g_3(\|x_n-(PT_1)^n y_n\|)$$
$$\le a_{n1}\|x_n-p\|^2+b_{n1}k_n^6\|x_n-p\|^2+c_{n1}k_n^6\|x_n-p\|^2$$
$$-(b_{n1}+c_{n1})\varepsilon^2 g_2(\|x_n-(PT_3)^n z_n\|)-\varepsilon^2 g_3(\|x_n-(PT_1)^n y_n\|)$$
$$\le k_n^6\|x_n-p\|^2-\varepsilon^2 g_3(\|x_n-(PT_1)^n y_n\|),$$

which implies that $g_3(\|x_n-(PT_1)^n y_n\|)\to 0$ as $n\to\infty$. Since $g_3:[0,\infty)\to[0,\infty)$ with $g_3(0)=0$ is a continuous strictly increasing convex function, it follows that

$$\lim_{n\to\infty}\|x_n-(PT_1)^n y_n\|=0. \tag{3.10}$$

Similarly, we have

$$\lim_{n\to\infty}\|x_n-(PT_2)^n y_n\|=0. \tag{3.11}$$

It follows from (1.5), (3.5) and (3.6) that

$$\|z_n-x_n\|=\|a_{n3}x_n+b_{n3}(PT_5)^n x_n+c_{n3}(PT_6)^n x_n-x_n\| \tag{3.12}$$
$$\le b_{n3}\|x_n-(PT_5)^n x_n\|+c_{n3}\|x_n-(PT_6)^n x_n\|$$
$$\to 0, \text{as} n\to\infty$$

It follows from (3.8) and (3.12) that

$$\|(PT_3)^n z_n-z_n\|\le\|x_n-(PT_3)^n z_n\|+\|z_n-x_n\| \tag{3.13}$$
$$\to 0, \text{as } n\to\infty$$

Similarly we have

$$\lim_{n\to\infty}\|(PT_4)^n z_n-z_n\|=0. \tag{3.14}$$

Noting that $y_n-z_n=a_{n2}(x_n-z_n)+b_{n2}((PT_3)^n z_n-z_n)+c_{n2}((PT_4)^n z_n-z_n)$, we have

$$\|y_n-z_n\|\le a_{n2}\|x_n-z_n\|+b_{n2}\|(PT_3)^n z_n-z_n\|+c_{n2}\|(PT_4)^n z_n-z_n\|$$

This with (3.12), (3.13) and (3.14) implies that

$$\lim_{n\to\infty}\|y_n-z_n\|=0 \tag{3.15}$$

From (3.12) and (3.15)

$$\lim_{n\to\infty}\|x_n-y_n\|=0 \tag{3.16}$$

It follows from (3.10) and (3.16) that

$$\left\|(PT_1)^n y_n - y_n\right\| \le \left\|x_n - (PT_1)^n y_n\right\| + \|x_n - y_n\| \to 0, \text{as} n \to \infty \tag{3.17}$$

Similarly we have

$$\lim_{n\to\infty}\left\|(PT_2)^n y_n - y_n\right\| = 0. \tag{3.18}$$

From (3.10) and (3.11), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \left\|a_{n1}x_n + b_{n1}(PT_1)^n y_n + c_{n1}(PT_2)^n y_n - x_n\right\| \\ &\le b_{n1}\left\|x_n - (PT_1)^n y_n\right\| + c_{n1}\left\|x_n - (PT_2)^n y_n\right\| \\ &\to 0, \text{as} n \to \infty\end{aligned} \tag{3.19}$$

Noting that

$$\begin{aligned}\left\|x_n - (PT_3)^n x_n\right\| &\le \|x_n - z_n\| + \left\|(PT_3)^n z_n - z_n\right\| + \left\|(PT_3)^n z_n - (PT_3)^n x_n\right\| \\ &\le (1+k_n)\|x_n - z_n\| + \left\|(PT_3)^n z_n - z_n\right\|\end{aligned}$$

This with (3.12) and (3.13) implies that

$$\lim_{n\to\infty}\left\|x_n - (PT_3)^n x_n\right\| = 0 \tag{3.20}$$

Similarly we have

$$\lim_{n\to\infty}\left\|x_n - (PT_4)^n x_n\right\| = 0 \tag{3.21}$$

Noting that

$$\begin{aligned}\left\|x_n - (PT_1)^n x_n\right\| &\le \|x_n - y_n\| + \left\|(PT_1)^n y_n - y_n\right\| + \left\|(PT_1)^n y_n - (PT_1)^n x_n\right\| \\ &\le (1+k_n)\|x_n - y_n\| + \left\|(PT_1)^n y_n - y_n\right\|\end{aligned}$$

This with (3.16) and (3.17) implies that

$$\lim_{n\to\infty}\left\|x_n - (PT_1)^n x_n\right\| = 0 \tag{3.22}$$

Similarly we have

$$\lim_{n\to\infty}\left\|x_n - (PT_2)^n x_n\right\| = 0. \tag{3.23}$$

Since an asymptotically nonexpansive mapping with respect to P must be uniformly L-Lipschitzian with respect to P ,where $L = \sup_{n\ge1}\{k_n\} \ge 1$, then we have

$$\begin{aligned}\|x_{n+1} - (PT_i)x_{n+1}\| &\le \left\|x_{n+1} - (PT_i)^{n+1}x_{n+1}\right\| + \left\|(PT_i)^{n+1}x_{n+1} - (PT_i)x_{n+1}\right\| \\ &\le \left\|x_{n+1} - (PT_i)^{n+1}x_{n+1}\right\| + L\left\|x_{n+1} - (PT_i)^n x_{n+1}\right\| \\ &\le \left\|x_{n+1} - (PT_i)^{n+1}x_{n+1}\right\| + L\|x_{n+1} - x_n\| + L\left\|x_n - (PT_i)^n x_n\right\|\end{aligned}$$

$$+L\left\|(PT_i)^n x_n-(PT_i)^n x_{n+1}\right\|$$
$$\le\left\|x_{n+1}-(PT_i)^{n+1}x_{n+1}\right\|+L\left\|x_n-(PT_i)^n x_n\right\|$$
$$+L(L+1)\|x_{n+1}-x_n\|.$$

Consequently, by (3.5), (3.6), (3.20), (3.21), (3.22), (3.23) and (3.19), it can be obtained that

$$\lim_{n\to\infty}\|x_n-(PT_i)x_n\|=0(i=1,2,3,4,5,6) \tag{3.24}$$

This completes the proof.

Theorem 1. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_i:K\to E$ $(i=1,2,3,4,5,6)$ be six nonself-asymptotically nonexpansive mappings with respect to P with common sequence $\{k_n\}\subset[1,\infty)$ satisfying $\sum_{n=1}^{\infty}(k_n-1)<\infty$. Suppose that $\{x_n\}$ is defined by (1.5), where $\{a_{ni}\}$, $\{b_{ni}\}$ and $\{c_{ni}\}$ $(i=1,2,3)$ are sequences in $[\varepsilon,1-\varepsilon]$ for some $\varepsilon\in(0,1)$ and $\overline{F}=\bigcap_{i=1}^{6}F(T_i)\neq\varnothing$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{6}$ if and only if $\liminf_{n\to\infty}d(x_n,\overline{F})=0$, where $d(x_n,\overline{F})=\inf\{\|x_n-p\|:p\in\overline{F}\}$.*

Proof. The necessity is obvious. Thus, we need only prove the sufficiency. Suppose that $\liminf_{n\to\infty}d(x_n,\overline{F})=0$. From (3.3), we obtain

$$d(x_{n+1},\overline{F})\le\left(1+(k_n^3-1)\right)d(x_n,\overline{F}). \tag{3.25}$$

As $\sum_{n=1}^{\infty}(k_n^3-1)<\infty$, therefore $\lim_{n\to\infty}d(x_n,\overline{F})$ exists by Lemma 1. But by hypothesis $\liminf_{n\to\infty}d(x_n,\overline{F})=0,$ hence we must have $\lim_{n\to\infty}d(x_n,\overline{F})=0.$ Next we shall prove that $\{x_n\}$ is a Cauchy sequence. It follows from (3.3) that for any $n,m\ge n_0$

$$\|x_{n+m}-p\|\le(1+\theta_{n+m-1})\|x_{n+m-1}-p\|$$
$$\le\exp(\theta_{n+m-1})\|x_{n+m-1}-p\|$$
$$\le\cdots\le\exp\left(\sum_{k=n}^{n+m-1}\theta_k\right)\|x_n-p\|.$$

Let $M=\exp\left(\sum_{k=1}^{\infty}\theta_k\right)$, then $M>0$ and

$$\|x_{n+m}-p\|\le M\|x_n-p\|,\ \forall n,m\ge n_0. \tag{3.26}$$

For an arbitrary $\varepsilon>0,$ since $\lim_{n\to\infty}d(x_n,\overline{F})=0$, there exists a positive integer N_1 such that $d(x_n,\overline{F})<\frac{\varepsilon}{4M}$ for all $n\ge N_1$. So, we have $d(x_{N_1},\overline{F})<\frac{\varepsilon}{4M}$. This means that there exists a $x^*\in\overline{F}$ such that $\|x_{N_1}-x^*\|\le\frac{\varepsilon}{4M}$. It follows from (3.26) that for all $n\ge N_1$ and $m\ge1$,

$$\|x_{n+m}-x_n\|\le\|x_{n+m}-x^*\|+\|x_n-x^*\|$$

$$\leq 2M\left\|x_{N_1} - x^*\right\|$$
$$< \varepsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since E is complete, therefore $\lim_{n\to\infty} x_n$ exists. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in K$. It remains to prove that $q^* \in \overline{F}$. For an arbitrary $\bar{\varepsilon} > 0$, there exists a positive integer $N_2 > N_1$ such that $\left\|x_n - q^*\right\| < \frac{\bar{\varepsilon}}{2(1+k_n)}$ for all $n \geq N_2$. Since $\lim_{n\to\infty} d\left(x_n, \overline{F}\right) = 0$, there exists a natural number $N_3 > N_2$ such that $d\left(x_n, \overline{F}\right) < \frac{\bar{\varepsilon}}{2(1+k_n)}$ for all $n \geq N_3$. Therefore, there exists $p^* \in \overline{F}$ such that $\left\|x_{N_3} - p^*\right\| < \frac{\bar{\varepsilon}}{2(1+k_n)}$. Consequently we have

$$\begin{aligned}
\left\|PT_1 q^* - q^*\right\| &= \left\|PT_1 q^* - p^* + \left(p^* - x_{N_3}\right) + \left(x_{N_3} - q^*\right)\right\| \\
&\leq \left\|PT_1 q^* - p^*\right\| + \left\|p^* - x_{N_3}\right\| + \left\|x_{N_3} - q^*\right\| \\
&\leq k_n\left\|q^* - p^*\right\| + \left\|p^* - x_{N_3}\right\| + \left\|x_{N_3} - q^*\right\| \\
&\leq (1+k_n)\left[\left\|p^* - x_{N_3}\right\| + \left\|x_{N_3} - q^*\right\|\right] \\
&< \bar{\varepsilon}.
\end{aligned}$$

This implies that $q^* \in F(PT_1)$. It follows from Lemma 2 that $q^* \in F(T_1)$. Similarly, $q^* \in F(T_2)$, $q^* \in F(T_3)$, $q^* \in F(T_4)$, $q^* \in F(T_5)$ and $q^* \in F(T_6)$. Therefore $q^* \in \overline{F}$. This completes the proof.

Theorem 2. *Let K be a nonempty closed convex subset of a real smooth and uniformly convex Banach space E with P as a sunny nonexpansive retraction. Let $T_i : K \to E$ $(i = 1,2,3,4,5,6)$ be six weakly inward and nonself-asymptotically nonexpansive mappings with respect to P with common sequence $\{k_n\} \subset [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by (1.5), where $\{a_{ni}\}$, $\{b_{ni}\}$ and $\{c_{ni}\}$ $(i = 1,2,3)$ are sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. If one of $\{T_i\}_{i=1}^{6}$ is completely continuous and $\overline{F} = \bigcap_{i=1}^{6} F(T_i) \neq \varnothing$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{6}$.*

Proof. By Lemma 4 $\lim_{n\to\infty}\left\|x_n - p\right\|$ exists for any $p \in \overline{F}$. It is sufficient to show that $\{x_n\}$ has a subsequence which converges strongly to a common fixed point of $\{T_i\}_{i=1}^{6}$. By Lemma 5, $\lim_{n\to\infty}\left\|x_n - (PT_i)x_n\right\| = 0$ $(i = 1,2,3,4,5,6)$. Suppose that T_1 is completely continuous. Noting that P is nonexpansive, we conclude that there exists subsequence $\{PT_1 x_{n_j}\}$ of $\{PT_1 x_n\}$ such that $PT_1 x_{n_j} \to p$, and hence $x_{n_j} \to p$ as $j \to \infty$. By the continuity of $P, T_1, T_2, T_3, T_4, T_5$ and T_6, we have $p = PT_1 p = PT_2 p = PT_3 p = PT_4 p = PT_5 p = PT_6 p$, and so $p \in \overline{F}$ by Lemma 2. Thus, $\{x_n\}$ converges strongly to a common fixed point p of $\{T_i\}_{i=1}^{6}$. This completes the proof.

Theorem 3. *Let K be a nonempty closed convex subset of a real smooth and uniformly convex Banach space E with P as a sunny nonexpansive retraction. Let $T_i : K \to E$ $(i = 1,2,3,4,5,6)$ be six weakly inward nonself-asymptotically nonexpansive mappings with respect to P with common sequence $\{k_n\} \subset [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by (1.5), where $\{a_{ni}\}$, $\{b_{ni}\}$ and $\{c_{ni}\}$ $(i = 1,2,3)$ are sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. if one of $\{T_i\}_{i=1}^{6}$ is demicompact and $\overline{F} = \bigcap_{i=1}^{6} F(T_i) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{6}$.*

Proof. Since one of $\{T_i\}_{i=1}^{6}$ is *demicompact, so is one of* $PT_1, PT_2, PT_3, PT_4, PT_5$ and PT_6. Suppose that PT_1 is *demicompact. Noting that* $\{x_n\}$ is bounded, we assert that there exists a subsequence $\{PT_1 x_{n_j}\}$ of $\{PT_1 x_n\}$ such that $PT_1 x_{n_j}$ converges strongly to p. By Lemma 5, we have $x_{n_j} \to p$ as $j \to \infty$. By the continuity of $P, T_1, T_2, T_3, T_4, T_5$ and T_6, we have $p = PT_1 p = PT_2 p = PT_3 p = PT_4 p = PT_5 p = PT_6 p$, and so $p \in \overline{F}$ by Lemma 2. By Lemma 4, we know that $\lim_{n\to\infty}\|x_n - p\|$ exists. Therefore, $\{x_n\}$ converges strongly to a common fixed point p as $n \to \infty$. This completes the proof.

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