Primal and Weakly Primal Sub Semi Modules

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Abstract

Let R be a commutative semiring with identity and M an R-semi module. The concept of primal sub modules has been introduced and studied [1].Also weakly primal sub modules have been studied [2]. Throughout this work, we define primal and weakly primal sub semi modules as a new generalization of prime sub semi modules. We show that they enjoy of many of the properties of prime sub semi modules.
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1. Introduction

Throughout R will be a commutative semiring with nonzero identity andM is an R-semimodule. Primal and weakly primal ideals over commutative emirings have been studied [4]. The concept of prime subsemimodules has been introduced by G. Yesilot, K. Orel and U. Tekir [3]. Throughout this workwe investigate various properties of primal and weakly primal subsemimodulesand their generalizations. We also explore the relationship between primaland weakly primal subsemimodules as well as the relationship between weaklyprimal subsemimodules over R and weakly primal subsemimodulesover MR_S ,where S is the set of all cancelable elements in R. Moreover we define P-primal subsemimodules and study its relations with prime subsemimodules ofsemimodules and other semimodule structures. Finally we establish aoneto-one correspondence between P-weakly primal subsemimodules of an R-semimoduleM and the PR_S -weakly primal subsemimodulesof MR_S .

2. Primal and Weakly Primal Subsemimodules

G. Yesilot ,K .Oreland U. Tekir[3], defined that asubset N of an R-semimodule M is called a subsemimodule of M if for n,n´∈ N and r ∈ R , n+n´ ∈ N and nr ∈ N ,an element r ∈ R is said to be prime to N if rm∈ N(with m ∈ M) implies that m ∈N,that is , ($N :_M r$) =N and N is prime sub-semimodule of the R-semimodule M if foreach r ∈ R and m ∈ M , rm∈ N impliesr ∈($N :_R M$)or m ∈ N. We will denote the set of all elements of R that are not prime to N by S(N).

Definition 2.1 A proper subsemimodule N of M is called a primal subsemimodule of M if S(N) form an ideal of R. Let N a primal subsemimodule and P=S(N) ,then we say that N is a P−primal subsemimodule of M.
Theorem2.2 Let R be a commutative semiring and M be an R-semimodule. If N is a P−primal subsemimodule of M,then P is a prime ideal.

ProofSuppose that r,s∈ R\P . We will show that rs$\notin$ P. If there is m ∈M such that (rs) m ∈ N , then r(sm) ∈ N , so sm∈($N :_M r$) =N . Since r is prime to N, sm∈ N implies m∈($N :_M s$)=N . This shows m ∈ N, that is rs is prime to N, so rs$\notin$P as required.
Proposition2.3 Let N be a subsemimodule of an R-semimodule M. If N is a P-primal subsemimodule of M,then $(N :_R M) \subseteq P$.

Proof Let $r \in (N :_R M)$ andr≠0(since0∈P).Since N is a proper subsemimodule of M, $(N :_R M)$ is a proper ideal of R, there existsM∈M\N, with rm∈ N. This shows that r is not prime to N, that is r∈ P.

Proposition2.4 Let R be a commutative semiring,M an R-semimodule N and K R-subsemimodules of M with K⊆ N and P an ideal of R . Then the following hold.

(i) I f K is a P−primal subsemimodule of N and S(N)⊆P , then K is a P−primal subsemimodule of M.
(ii) Let (R,P) be a local semiring . If K is a P−primal subsemimodule of N and N is a primal subsemimodule of M , then K is primal subsemimodule of M.

Proof (i) If r ∈ R is not prime to K ,then there existssm∈ M\K with rm∈ K. If m ∈N ,then r∈ P . So suppose that m∉N .Therefore ,r ∈ S(N)⊆ p . Now let r ∈P , we will show that that r is not prime to K . There exist sn∈ N\K such that rn∈K .Thus n∈M\K gives r is not prime to K. Thus K is a P−primal subsemimodule of M.
(ii)This follows from (i).

Let R be a given semiring, S be the set of all multiplicatively cancelable elements of R and M bean R-semimodule. Then the semimodulegenerated by M in R_S , is the set of all finite sum sm1s1+m2s2+...+mnsn , where mi ∈ M and si∈ R_S ,which is denoted by MR_S .If N is a subsemimoduleof MR_S ,

define N∩M={m∈M; $\bar{1}$ ∈N}. Clearly N ∩ M is a subsemimodule of M.
G. Yesilot , K . Oreland U. Tekir [3], defined that the k-closure of a sub-semimodule N of R-semimodule M is N={ $\bar{a}$ ∈ M : $\bar{a}$ +a1=a2, for some ai∈ N, i=1,2} and a subsemimodule N is called a k-subsemimodule if N equals its k-closure.

Lemma2.5 Assume that R is a semiring .Let M be R-semimodule and N be a subsemimodule of the R_S-semimodule MR_S .Then the following hold:

(i) X ∈MR_S if and only If it can be written in the form $x = \frac{m}{c}$ for some m ∈M and c ∈ S.

(ii) (N ∩ M) R_S = N .

(iii) If N is prime subsemimodule of MRS then N ∩ M is prime subsemimodule of M.

Proof (i) Let x ∈MR_S .Then there are elements

s1 ∈R_S and mi ∈ M such that $x = \sum_{i=1}^{n} misi$

,withsi = $\frac{ri}{ci}$ where ri∈ R and ci∈ S .Put =c1c2...cn for suitable elements e1,e2,...,en ∈ R , we have si = $\frac{ei}{c}$

,therefore x= $\sum_{i-1}^{n} mi \cdot \frac{ei}{c} = \frac{m1.e1+...+mnen}{c} = \frac{m}{c}$,where m ∈ M.

The Converse Implication is Obvious.

(ii)(N∩M) R_S ⊆ N is trivial. For the reverse inclusion, ifx∈ N, then x = $\frac{m}{s}$, where m =m1e1 + m2e2 + … + mnen ∈ M and $s \in S$, so m=xs∈N∩ M and accordingly , x=m . $\frac{1}{s} \in R_S$.

(iii)Assume that N is prime subsemimodule of MR_S and rm∈ N ∩ M , N is a prime subsemimodule,then $(N :_{R_S} MR_S)$ or $\frac{m}{1}$ ∈ N, that is r ∈$(N \cap M :_R N)$ or $m \in N \cap M$.

Theorem2.6 LetM be an R-semimodule and S the set of multiplicatively cancelable subsets of R. If K is a Q−primal subsemimodule of MR_S ,then K∩M is Q∩ R− primal subsemimodule of M.

Proof Let N=K∩M and P=Q∩R .It is enough to show that S(N)=P.First ,let a ∈ S(N). There existssm∈ M\N such that am ∈ N. Then form $\frac{a}{1}\cdot\frac{m}{1}=\frac{am}{1}\in K$ and $\frac{m}{1}\in MR_S\setminus K$ we get $\frac{a}{1}\in$Q ,hence $a\in p$. On the other hand ,for every $r\in p$,we have $\frac{r}{1}\in Q$, so there exists $\frac{m}{s}\in R_S\setminus K$ such that $\frac{r}{1}\cdot\frac{m}{s}=\frac{rm}{s}\in K$

Then by lemma 2.5 (ii), $rm\in K\cap M$, so $\frac{r}{1}\cdot\frac{m}{1}=\frac{rm}{1}\in K$ with $\frac{m}{1}\in MR_S\setminus k$, $m\in M\setminus N$. So $rm\in K\cap N$ with $m\in M\setminus(K\cap M)$ This shows that r is not prime to N, that is $r\in S(N)$.

Definition 2.7 Let R be commutative semiring and M be an R-subsimemodule. A proper subsemimodule N of M is called a weakly prime subsemimodule if whenever 0≠rm ∈ N, for some r ∈ R and m ∈ M , then either m ∈ N or rM⊆ N.

Let R be a commutative semiring and N be a subsemimodule of an R- semimodule M. An element r ∈ R is called weakly prime to N if 0 ≠ rm∈N(m∈M) implies that m ∈ N . Otherwise r is not weakly prime to N.Denote By w(N) the set of elements of R that are not weakly prime to N. Let N be a subsemimodule of the R-semimodule M. Then 0 is always weakly prime to N and if r ∈ R is primeto N,then r is weakly prime to N, but the convers is not necessarily true.

Definition2.8 Let R be a commutative semiring with non -zero identity and let N be a propersubsemimodule of an R-semimodule M . N is called a weakly primal if the set P=w(N)∪{0} forms an ideal of R . P is called the (weakly) adjoint ideal of N and we also say N is P−weakly primal subsemimodule of M.

We next give several characterizations of weakly primal subsemimodules.

Theorem2.9 Let P be an ideal of a commutative semiring R,M an R- semimodule and N a subsemimodule of M. Then the following areequivalent :

(i) N is a P−weakly primal.

(ii) For every r ∉ P − {0}, $(N:_M r)=N\cup(0:_M r)$; and for every 0 ≠ r∈P, $N\cup(0:_M r)\not\subset(N:_M r)$.

Proof$(i)\Rightarrow(ii)$ Suppose that N is P−weakly primal subsemimodule of M. Let r∉ P − {0} = w(N) and let $m\in(N:_M r)$. If rm = 0 then $m\in(0:_M r)$. If rm ≠ 0, since r is weakly prime to N, we get $m\in N$. So $m\in N\cup(0:_M r)$, since $N\cup(0:_R r)\subseteq N$ for any subsemimodule, we have $(N:_M r)=N\cup(0:_M r)$.

Now suppose that $r\in$ P − {0} = w(N), r is not weakly prime to N, sothere exists m∈ M\N such that 0 ≠ rm∈ N , $m\in(N:r)\setminus(N(0:_M r))$

$(ii)\Rightarrow(i)$ It follows from (ii) that w(N) = P −{0}. Hence N is P−weakly primal.

Theorem 2.10 Let R be a commutative semiring, M be an R-semimodule, and N isa subsemimodule of M if N is P−weakly primal subsemimodule of Mthen P is weakly prime ideal of R.

Proof Suppose that a, b∈ R −P with ab ≠ 0 we will show that ab ∉ P. If there is m ∊ M with 0 ≠ (ab)m∊ N, then $0\neq am\in(N:_M b)=N\cup(0:_M b)$ by Theorem 2.9, where $am\notin(0:_M b),0\neq am\in M$. As a ∉ P, then a is weakly prime to N, hence m ∊ N that is ab is weakly prime to N, ab ∉ P.

Consequently P is weakly prime ideal of R.

Let R be commutative semiring. An R-semimodule M is called faithful(resp., cyclic) if Ann(M) =0, that is $(0:_R M)=0$(resp., M can be generated bya single element i.e; M = (x) = Rx).

Proposition 2.11 Let N be a faithful k-subsemimodule of an R-semimoduleM. If N is P−weakly primal subsemimodule of M, then $(N:_R M)\subseteq P$.

Proof Assume that N is P−weakly primal. Since N is a proper subsemimodule of N, then$(N:_R M)$is a proper ideal of R, there exists m ∊ M\N. Let r non- zero element r ∊$(N:_R M)$, then rm∊ N. If rm ≠ 0 and m ∉ N give that r is not weakly prime to N, that is r ∊ P.

So we assume that rm = 0 as N faithful subsemimodule there exists n ∈ N with rn ≠ 0.Now r(n + m) ∈ N with m+ n ∉ N since N is k-subsemimodule this implies that r is not weaklyprime to N, that is r ∈ P. Thus $(N :_R M) \subseteq$ P as required.

Theorem 2.12 Let R be commutative semiring, N is k-subsemimodule of faithful cyclic R-semimodule M, If N is P−weakly primal of M then $(N :_R M)$ is P−weakly primal ideal of R.

ProofLet M = Rx for some x ∈ M, and set I = $(N :_R M)$. We will show that w(I) = w(N). For every r ∈w(I), there exists a ∈ R\I such that 0 ≠ ra∈ I, rax ≠ 0 otherwise $ra \in (0 :_R x) = (0 :_R M) = 0$, since M is a faithfulas ax∈ M\N, it follows that r is not weakly prime to N, that is r ∈ w(N).

Now assume that r ∈ w(N), then 0 ≠ rm∈ N for some m ∈ M\N, we can write m = r´x for some r´∈ R , 0 ≠ rr´x∈ N, 0 ≠ rr´∈ I with r´∈ R\I , this shows r is not a weakly prime to I. Hence r ∈w(I),w(I) = w(N) as required.
We next give conditions for weakly primal subsemimodules to be primal subsemimodules.

Theorem 2.13 Let R be a commutative semiring and P is k-ideal of R. If a k-subsemimodule N of R-semimoduleM is P−weakly primal of M with $(N :_R M) \subseteq$ P and N$(N :_R M)$≠0. Then N is a primal subsemimodule of M.

ProofIt is enough to show that P = S(N). For every 0 ≠ r ∈ P, r is not a weakly prime to N, so r is not a prime to N, that is r ∈ S(N). Now assume that a ∈S(N). There exists m ∈ M\N such that am ∈ N. If am ≠ 0 thena is not a weakly prime to N that is a ∈ P. So assume that am = 0. **Case(1)**suppose that aN ≠ 0, say an_0 ≠ 0 for some n_0 ∈ N. Now 0 ≠ a(m + n_0) ∈ Nwith m + n ∈ M\N, implies that a is not weakly prime to N, and hence a ∈ P. **case(2)** suppose that aN = 0. If m$(N :_R M)$≠ 0, then rm ≠ 0 for somer ∈$(N :_R M)$. Now 0 ≠ (a + r)m∈ N with m ∈ M\N implies that a + r is not weakly prime to N, that is a + r ∈ P and hence a ∈ P since P is k-ideal.If m$(N :_R M)$= 0 since N$(N :_R M)$≠ 0, there is r1 ∈ (N :R M) and n1 ∈ N with r1n1 ≠ 0. Now 0 ≠ (a+r1)(m+n1) ∈ N with m+n1 ∈ M\N shows that a+r1 ∈ P with r1 ∈ P implies a ∈ P, since P is k-ideal. Therefore S(N) ⊆ Pand so S(N) = P which implies that N is P-primal.

Theorem 2.14 Let R bea commutative semiring and M be an R-semimodule.
Then a faithful k−weakly prime subsemimodule N of M is weakly primal.
Proof Let N be k−weakly prime subsemimodule of M with $Ann_R(N) = 0$.

Set P = $(N :_R M)$ as N a proper subsemimodule so that $(N :_R M)$ for every r ∈ P\{0},there exists m ∈ M\N such that rm∈ N. If rm ≠ 0, then r ∈w(N).

So we can assume rm = 0 as N is faithful subsemimodule, there exists n ∈ N with rn ≠ 0, 0 ≠ r(n + m) ∈ N with n + m ∈ M\N , implies that r is not weakly prime, that is r ∈ w(N). Now assume that r ∈w(N), there exist m ∈ M\N with 0 ≠ rm∈ N. As N is weakly prime we get r ∈$(N :_R M)$\{0}, hence w(N) ⊆ P\{0}. We have already show that P = w(N)∪{0}. Hence N is P−weakly primal.

Theorem 2.15 Let R be commutative semiring, S be the set of multiplicatively cancelableelemants of R, and N k-subsemimodule of M. If N is P−weakly primal of M with P S = . Then the following hold.

(i) If 0 ≠ m/s ∈ NR_S, then m ∈ N.

(ii) $(N :_R M)_{RS}$ = $(NR_S :_{RS} MR_S)$.

Proof (i) Suppose that 0 ≠ m/s ∈ NRS and m ∈ M\N, then there exists n ∈ N and t ∈ S with $\frac{m}{s} = \frac{n}{t}$, 0 ≠ tm = sn with m ∉ N, hence t is not a weaklyprime to N which is a contradiction.

(ii) Let $\frac{r}{s} \in (NR_S :_{RS} MR_S)$ for every m ∈ M, $\frac{r}{s} \cdot \frac{m}{1} = \frac{rm}{s} \neq \frac{0}{1} \in NR_S$, by (i) rm∈ N.

This implies that r $(N :_R M)$, that is $\frac{r}{s} \in (N :_R M)R_S$ and hence $(NR_S :_{RS} MR_S) \subseteq (N :_{RS} M)R_S$. The other containment is obvious.

Proposition 2.16 Let R be a commutative semiring, S the set of multiplicatively cancelable elements of R and N be k−subsemimodule of R−semimoduleM. If N is P−weakly primal with $P \cap S = \phi$ then the following hold.

(i) NR_S is PR_S −weakly primal subsemimodule of R_S -semimodule MR_S.

(ii) N = $NR_S \cap$M.

Proof (i)Let 0 ≠ $\frac{a}{s} \in PR_S$ then by Theorem 2.10, P is a weakly prime ideal and 0 ≠ a ∊ P by [4, lemma 8]. There exists m ∊ M\N such that 0 ≠ ma ∊ N, as we must have $0 \neq \left(\frac{m}{1}\right)\left(\frac{a}{s}\right) \in NR_S$, by proposition 4.26 $\frac{m}{1} \notin NR_S$, then $\frac{a}{s}$ is not weakly prime to NR_S, so $PR_S \backslash\{0\}$ ⊆ w(N). Now suppose that $\frac{r}{s}$∊ w(NR_S), then 0 ≠ $\frac{r}{s} \cdot \frac{m}{t} \in NR_S$ for some $\frac{m}{t} \in MR_S \setminus NR_S$, then 0 ≠ $\frac{rm}{st} \in NR_S$, by Proposition 2.15, 0 ≠ rm∊ N with m ∊ M\N , this shows r ∊ P\{0}, then $\frac{r}{s} \in PR_S \backslash\{0\}$.

(ii) Let m ∊ $NR_S \cap$M. If m = 0 then m ∊ N. If m ≠ 0, then 0 ≠ $\frac{m}{1} \in NR_S$ by Proposition 2.15 m ∊ N. The other containment is obvious.

Theorem 2.17 Let M be an R-semimodule and S the set of multiplicatively cancelable elements of R. If K is Q−weakly primal subsemimoduleof MR_S, thenK ∩ M is a (Q ∩ R)−weakly primal subsemimodule of M.

Proof Set N = K ∩ M and P = Q ∩ R. By [4, propsition 3.10], P is a weakly prime ideal of R. It is enough to show that w(N) = P\{0}. For everya ∊w(N) there exists m ∊ M\N such that 0 ≠ am ∊ N, $\frac{0}{1} \neq \frac{a}{1} \cdot \frac{m}{1} = \frac{am}{1}$ ∊ Kand $\frac{m}{1} \in MR_S$ \K that $\frac{a}{1}$ ∊ Q\{0} so a ∊ P\{0}. This shows w(N) ⊆ P\{0}.

Now let a ∊ P\{0}, then $\frac{a}{1}$ ∊ Q\{0} as K is Q−weakly primal subsemimodule of MR_S, there exists $\frac{m}{s} \in MR_S$ \K with $\frac{0}{1} \neq \frac{a}{1} \cdot \frac{m}{s} = \frac{am}{1}$ ∊ K, then 0 ≠ $\frac{am}{s}$ ∊ K,0 ≠ am ∊ N with m ∊ M\N. Thus a ∊w(N), that is P\{0} ⊆ w(N) as required.

Theorem 2.18 Let P be a weakly prime ideal of commutative semiring R and let M be an R-semimodule. Assume that S is multiplicatively cancelable subset of R with P S = . Then there exists a one-to-one correspodence betweenthe P-weakly primal subsemimodules of M and the PRS−weakly primal subsemimodules of MRS.

Proof This follows from Proposition 2.16 and Theorem 2.17.

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